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Geometría de Finsler: Espacios de Landsberg

Memorias del Primer Taller Internacional sobre Geometría semi-Riemanniana y de Finsler.

Universidad Autónoma de San Luis Potosí. México. 24-26 de mayo de 2006.

La Universidad Autónoma de San Luis Potosí organizó el Primer Taller Internacional sobre Geometria Semi-Riemanniana y de Finsler, que se llevó a cabo en la Universidad Autónoma de San Luis Potosí, México, del 24 al 26 de mayo de 2006.

El propósito de este taller fue el de proveer un primer acercamiento a la Geometria semi-Riemanniana y de Finsler a matemáticos y estudiantes que, posiblemente, no estaban familiarizados con ellas. Para este fin, se invitó a dos expertos para dar dos cursos cortos intensivos, en los temas referentes al título. Los cursos fueron complementados con pláticas de otros expertos en la materia.

Los comités estuvieron conformados como sigue:

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A short review on Landsberg spaces*

C.T.J. Dodson
School of Mathematics, University of Manchester
Manchester M60 1QD, UK

Abstract

This short review is concerned with real finite-dimensional Finsler manifolds (M, F) with Finsler structures $F : TM \to [0, \infty)$ that satisfy the Landsberg conditions. In particular this includes the case of Berwald manifolds since their Chern connections on π^*TM are fibre-independent. The aim is to provide an annotated collection of references to geometric results that seem important in the study of Landsberg spaces and to suggest some areas for further work in this context.

AMS Subject Classification (2001): 53B40, 53C60, 58B20
Key words: Finsler manifold, Cartan tensor, Ehresmann connection, Chern connection, curvature, exponential map, Landsberg space, Berwald space, Randers space, conformal equivalence, Killing field, parallel isometry. completeness

1 Introduction

Finsler manifolds (or spaces) can be thought of as generalizations of Riemmanian manifolds; tangent spaces carry Minkowski norms instead of inner products and geometric objects on tangent vectors depend not only on the base but also on the fibre component. Chern and Shen [26] have provided an authoritative treatment of the subject, for which treatise the book by Bao, Chern and Shen [20] is a helpful introduction with a wealth of detail and we mainly follow the notational conventions of these authors and, similarly, we restrict our attention only to the real case. Finsler manifolds have intrinsic geometrical significance and also they have been used to model a variety of problems from dynamics, optics, ecology and relativity, cf. eg. Antonelli et al. [4], Bao et al. [20] and Asanov [7].

Consider $\phi : U :\to \mathbb{R}^n : x \mapsto (x^i)$ as a local coordinate system on an open set U of a $\mathbb{C}^\infty$ manifold M, with (∂_{x^i}) as the induced coordinate basis for the tangent space T_xM at a point $x \in M$. Let

$$F : TM \to [0, \infty) : (x, y) \mapsto F(x, y)$$

be $\mathbb{C}^\infty$ on $TM \setminus \{0\}$, positively homogeneous of degree 1 in the fibre coordinate, and satisfying for each $x \in M$ the Minkowski norm condition

$$\lim_{s\to 0, t\to 0} \frac{F^2(x, y + su + tv)}{2} = g_y(u, v)$$

where g_y is an inner product on T_xM. In this case, using local coordinates for $(x, y) \in TM \setminus \{0\}$, the Hessian

$$[g_{ij}] = [\partial_{y^i}\partial_{y^j}(\frac{F^2(x, y^i\partial_{y^i})}{2})] \tag{1.1}$$

is positive definite, so as a matrix it has everywhere rank $n-1$. Then we call F a Finsler structure on TM and, at each $x \in M$, $F(x, -)$ is a Minkowski norm on T_xM. Given a manifold M and a Finsler structure F on TM, the pair (M, F) is called a Finsler manifold. Sometimes, an n-dimensional Finsler manifold (M, F) is referred to as F^n with base space M. In the circumstance that each $F(x, -)$ is a

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Euclidean norm, subordinate to an inner product on T_xM, we reduce to the Riemannian case (M, g) with

$$F(x^i, y^j) = g_{ij} y^i y^j.$$

On a Finsler manifold (M, F), the family of Minkowski norms $\{F(x, -)|x \in M\}$ yields a length function on oriented piecewise smooth curves in M. By taking infima over all such curves between a given pair of points in M we obtain a function d_F that is positive definite and satisfies the triangle inequality but it is not necessarily symmetric. The topology induced by d_F coincides with the manifold topology of M.

The matrix (1.1) gives the components of the fundamental tensor g of (M, F); explicitly, g is a Riemannian metric tensor on the pullback bundle π^*TM (over $TM \setminus \{0\}$) in natural (pullback) coordinates denoted also by (x^i). This Riemannian metric on π^*TM determines formal Christoffel symbols, γ^i_{jk} via the usual formula.

The Cartan tensor A is a trilinear symmetric form on π^*TM (over $TM \setminus \{0\}$) and has components

$$[A_{ijk}] = \frac{F}{2}[\partial_{y^k} g_{ij}] = \frac{F}{4}[\partial_{y^i}\partial_{y^j}\partial_{y^k} F^2]. \tag{1.2}$$

For Riemannian manifolds, $A = 0$.

The Hilbert form ω on π^*TM (over $TM \setminus \{0\}$) and its dual ℓ are given in natural coordinates by

$$\omega = \partial_{y^i} F \; dx^i = \omega_i \; dx^i \tag{1.3}$$

$$\ell = \frac{y^i}{F} \; \partial_{x^i} = \ell^i \; \partial_{x^i} \tag{1.4}$$

and they satisfy $\omega(\ell) = 1 = g(\ell, \ell)$.

The nonlinear Ehresmann connection on π^*TM has components N^i_j given by

$$\begin{aligned} N^i_j &= \gamma^i_{jk} y^k - \frac{1}{F} g^{im} A_{mjk} \gamma^k_{rs} y^r y^s \\ &= \gamma^i_{jk} y^k - C^i_{jk} \gamma^k_{rs} y^r y^s. \end{aligned} \tag{1.5}$$

The Chern (linear torsion free) connection ∇ on π^*TM has components Γ^i_{jk} given by

$$\begin{aligned} \Gamma^i_{jk} &= \gamma^i_{jk} - \frac{g^{im}}{F}\left(A_{mjs} N^s_k - A_{jks} N^s_m + A_{kms} N^s_j\right) \\ &= \frac{g^{is}}{2}\left(\delta_{x^k} g_{sj} - \delta_{x^s} g_{jk} + \delta_{x^j} g_{ks}\right) \end{aligned} \tag{1.6}$$

with connection forms ω^i_j satisfying

$$\omega^i_j = \Gamma^i_{jk} dx^k \tag{1.7}$$

$$d(dx^i) - dx^j \wedge \omega^i_j = -dx^j \wedge \omega^i_j = 0 \tag{1.8}$$

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = \frac{2}{F} A_{ijs} \delta_{y^s}. \tag{1.9}$$

If the Γ^i_{jk} are functions of x only, then (M, F) is called a Berwald space (or manifold) and F is called a Berwald metric; in this case the Γ^i_{jk} actually arise from a family of associated Riemannian metrics. The Finsler manifold is called a Landsberg space (or manifold) and F is called a Landsberg metric if

$$\ell^s \nabla_{\delta^s_x}(A_{ijk}) = 0 \tag{1.10}$$

$$\text{equivalently } \; 0 = \left(\delta_{x^s} A_{ijk} - A_{mjk}\Gamma^m_{is} - A_{imk}\Gamma^m_{js} - A_{ijm}\Gamma^m_{ks}\right)\ell^s \tag{1.11}$$

$$\text{equivalently } \; \Gamma^i_{kj} = \frac{1}{2}\partial_{y^j}\partial_{y^k}(\Gamma^i_{kj} y^k y^j). \tag{1.12}$$

Particular Finsler metric functions include:

(i) $F = \alpha + \beta$, the Randers metric,

(ii) $F = \alpha^2/\beta$, the Kropina metric,
where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric, and $\beta = b_i(x)y^i$ is a non-zero differential 1-form on M. A Randers Landsberg space of dimension two is a Berwald space; a Kropina space of dimension two with $b^2 = 0$ is a Landsberg space, then it is a Berwald space (where $b^2 = a_{rs}(x)b^r b^s$ and a_{rs} is the associated Riemannian metric tensor) [34].

A Finsler manifold is called locally projectively flat if, for all $(x, y) \in TM \setminus \{0\}$,

$$y^k \partial_{x^k} \partial_{y^i} F(x, y) = \partial_{x^i} F(x, y) \tag{1.13}$$

then about every point there are local charts in which the geodesic segments are mapped to lines in coordinate space. A Berwald space with scalar curvature is projectively flat and it is Riemannian if of nonzero constant curvature.

Atkin [11] proved a very nice theorem that extends to the Finsler case the Hopf-Rinow completeness theorem of Riemannian manifolds. He proved that a connected C^1 manifold M of finite dimension, possibly with boundary, admits a bounded complete Finsler structure if and only if it is compact. Furthermore, if M is a C^1 Banach manifold admitting a complete Finsler structure, and N is a connected noncompact C^1 Banach manifold admitting a bounded complete Finsler structure, then $M \times N$ admits a bounded complete Finsler structure. It follows that any C^1 Banach manifold satisfying a certain stability condition admits a complete bounded Finsler structure. See also Bao et al. [20] Chapter VI for a detailed study of Finsler forward and backward completeness and reference there to the thesis work of Dazord in 1969.

Rademacher [80, 81] extended some classical comparison theorems in Riemannian geometry to the Finsler case. He introduced the reversibility

$$\lambda := \max\{F(x, -y) \mid F(x, y) = 1\},$$

and showed that if M is a simply-connected, compact Finsler manifold of dimension $n \geq 3$ with reversibility λ and the flag curvature satisfies $(1 - \frac{1}{1+\lambda})^2 < K \leq 1$, then the length of a geodesic loop is at least $\pi(1 + \frac{1}{\lambda})$. He then proved that a simply-connected and compact Finsler manifold of dimension $n \geq 3$ with reversibility λ and flag curvature $(1 - \frac{1}{1+\lambda})^2 < K \leq 1$ is homotopy equivalent to the n-sphere.

1.1 Examples

We collect some examples but see the following texts for details of many more: Bao, Chern and Shen [20], Matsumoto [64], Shen [86], Chern and Shen [26], Antonelli et al. [4, 6] and Asanov [7].

1. A Finsler manifold (M, F) is called a Finsler torus if the space is homeomorphic to a torus, and a two-dimensional Finsler torus is called flat if it is equipped with a Finsler structure that is obtained by passing to a quotient from a Finsler structure of $\mathbb{R}^2$ invariant under translations.

2. If a two-dimensional Finsler torus is a Landsberg space and has no conjugate point then the space is isometric to a flat Finsler torus [28].

3. 3-dimensional Landsberg spaces of constant curvature are either Riemannian spaces or spaces of vanishing curvature [68]. Numata [70] showed that every n-dimensional Landsberg metric of scalar flag curvature $K \neq 0$, $(n > 2)$ is Riemannian with constant curvature. It will be Riemannain if $K = 0$.

4. In a Finsler space of two dimensions the holonomy group is in general an infinite continuous group. This group has one, two or three parameters but this reduces to one parameter for a Landsberg space [68].

5. All non-Riemannian Berwald spaces can be constructed from cartesian products among 54 basic non-Riemannian Berwald spaces devised by Szabo [92].

6. A Finsler space of dimension 2 is a generalized Berwald space if and only if the first derivative of the main scalar by the Landsberg angle gives a differential equation of the form $y = f'(y)$.

7. Lee and Park [55] showed that a Finsler space (M, F) with (α, β)-metric

$$L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha} = \beta \sum_{r=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^r, \ \alpha = (a_{ij}(x)y^i y^j)^{1/2}, \beta = b_i(x)y^i \tag{1.14}$$

is projectively flat if and only if it is a Berwald space and the associated Riemannian space is projectively flat. Shen [87] pointed out that for locally projectively flat Berwald metrics, the non-trivial case is in dimension two, because locally projectively flat metrics are of scalar flag curvature and Berwald metrics are Landsberg metrics. Bacso-Matsumotos result [12] is a consequence of Numata [70] in dimension greater than two.

8. Bácsó and Matsumoto [16] showed that a Finsler n-manifold (M, F) is a projectively flat Berwald space if and only if it belongs to one of the following classes: (1) $n \geq 3$: (a) locally Minkowski spaces, (b) Riemannian spaces of constant curvature; or (2) $n = 2$: (a) locally Minkowski spaces, (b) Riemannian spaces of constant curvature, (c) spaces with $L = \beta^2/\gamma$ and the signature $\epsilon = +1$.

2 Landsberg Geometry

Every Berwald space is a Landsberg space; however, it has been a long-standing problem to decide if the converse is true [85]. Vattamány [99] proved the general case that every Landsberg space with vanishing Douglas tensor is a Berwald space. Hence, a Finsler manifold (M, F) is a Berwald space if there exists a symmetric linear connection ∇ having parallel translation which preserves the metric; if such a connection ∇ is flat, then (M, F) is said to be locally Minkowski [74]. Muzsnay [69] has shed more light on the equivalence problem by establishing conditions under which a second-order homogeneous ordinary differential equation (spray) be Finsler metrizable—ie the geodesic equation of a Finsler space, in terms of the holonomy algebra generated by horizontal vector-fields. Muzsnay obtained similar results for the Landsberg case, in particular, he proved that a quadratic second order differential equation is Landsberg metrizable if and only if it is Finsler metrizable. Asanov [10] suggests some new methodology by his introduction of the concept of Finsleroid Finsler spaces. Bao [21] elaborated the details of Asanov's new examples and showed that there are indeed Landsberg spaces which are not Berwald, albeit containing a singularity at some $y \in T_xM$. Shen [88] has since provided a new study of Landsberg (α, β) metrics and shows that a regular such metric is Landsberg if and only if it is a Berwald metric; relaxing regularity allowed him to construct a family of Landsberg metrics that are not Berwald.

Mo [67] gave geometric and algebraic characterizations for Finsler spaces with zero Riemann curvature. In particular, such spaces are characterized by the fact that the horizontal distribution of the projective sphere bundle has a flat foliation.

Wang [102] solved the problem of determining all Finsler spaces of dimensions $n > 2$ which admit a $\frac{1}{2}n(n + 1)$ parameter group of motions: they are the Riemann spaces of constant curvature. The method of proof uses results on Lie groups and linear groups rather than the classical method of studying integrability conditions. Singh et al. [90] obtained conditions for a vector field to be a Killing field in a Randers space. Killing field properties in Finsler manifolds have been studied by Yawata [105] who gave the form of Killing equations with respect to the Cartan and Berwald connections. Lovas [56] studied Killing fields whose integral curves are geodesics of an associated Finsler manifold, including the case of a Randers metric.

Antonelli and Lackey [5] and Antonelli and Zastawniak [6] provided detailed treatments of the Finsler analogue of Laplace operators and Hodge decomposition, and the classification theorem for 2-dimensional Berwald spaces which are not locally Minkowski. Centore [24] provided a very elegant characterisation of Riemannian spaces as a subset of Finsler spaces and Berwald spaces as a subset of Finsler spaces, solely in terms of the two naturally associated volume forms: Riemannian and Busemann [23]. For further discussion of the role of the Busemann volume form, in the context of developing a Laplacian for Finsler spaces, see Centore [25].

For a positive definite Finsler manifold the associated Levi-Civita connection coincides with the canonical connection if the Finsler space reduces to a Berwald space [92] and if a linear connection on the base manifold is compatible with the horizontal distribution of a Finsler space, then it is compatible with respect to the associated Riemannian metric [100].

A Berwald space (M, F) is locally, respectively globally, symmetric if the Chern connection is locally, respectively globally, symmetric. Deng [29] proved that every locally symmetric Berwald symmetric space is locally isometric to a globally symmetric Berwald space; this extends to locally geodesic Berwald spaces.

Yang [103] reported a necessary and sufficient condition for a Finsler space with (a, β)-metric to be a Berwald space, and studied the conformal changes between two (a, β)-metric Finsler spaces.

Finsler manifolds admit exponential maps and normal neighbourhoods. In particular, Kristaly et al. [51] proved that in a Berwald space (M, F) of non-positive curvature, every point $x \in M$ admits a neighborhood, such that two geodesics $\gamma_1, \gamma_2 \colon [0, 1] \to M$, emanating from x ($\gamma_1(0) = \gamma_2(0) = x$), satisfy the inequality

$$2d_F(\gamma_1(\tfrac{1}{2}), \gamma_2(\tfrac{1}{2})) \leq d_F(\gamma_1(1), \gamma_2(1)). \tag{2.15}$$

It follows that the length of a median of a geodesic triangle in M is smaller than or equal to the length of the corresponding side.

Lee and Park [55] studied Finsler spaces (M, F) with (α, β)-metric

$$L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha} = \beta \sum_{r=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^r, \ \alpha = (a_{ij}(x)y^i y^j)^{1/2}, \beta = b_i(x)y^i \tag{2.16}$$

They proved that (M, F) is a Berwald space if and only if $b_{j;i} = 0$; then the Berwald connection is Riemannian. (M, F) is projectively flat if and only if it is a Berwald space and the associated Riemannian space is projectively flat. Lee [53] studied a Finsler space with the special (α, β) metric

$$L(\alpha, \beta) = c_1\alpha + c_2\beta + \alpha^2/\beta$$

satisfying some conditions, finding a condition under which this special Finsler space is a Berwald space. If a two-dimensional Finsler space with this metric $L(\alpha, \beta)$ is a Landsberg space, then it is a Berwald space.

Park and Lee [77] considered a Finsler space $F^n = (M, L)$ with a generalized Randers metric

$$L^2(\alpha, \beta) = c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2$$

(where $\alpha^2 = a_{ij}(x)y^i y^j$ may be any quadratic form, $\beta = b_i(x)y^i$ and c_1, c_2, c_3 are nonzero constants). This type of Finsler space is a Landsberg space if and only if b_i is a Killing vector field with constant length, and such a Landsberg space is a Berwald space.

Following the preprint of Asanov [10], Shen [88] has provided a new study of Landsberg (α, β) metrics and shows that a regular such metric is Landsberg if and only if it is a Berwald metric.

Tamássy [94] studied the class of Finsler spaces that admit metrical linear connections; these are exactly the affine deformations of the associated locally Minkowski spaces. Moreover, a Finsler space admits a metrical linear connection in the tangent bundle TM if and only if it is an affine deformation of a Berwald space with vanishing h-curvature tensor K of its Rund connection.

Pandey and Tiwari [73] studied the Landsberg and semi-C-reducible Landsberg cases and explicitly derived expressions of the h-connection vectors and of the h-covariant derivative of the h-hv torsion tensor. They thereby obtained necessary and sufficient conditions for a 4-dimensional semi-C-reducible Landsberg space to be a Berwald space, in terms of relations satisfied by the nontrivial main scalars.

Dragomir [30] reported the following results on harmonic maps: Let $(M^n(c), E)$ be a Finsler space of scalar curvature $c \neq 0$ and vanishing mixed torsion vector $P_j = \dot{\partial}_i N^i_j - F^i_{ji}$. All h-harmonic functions $f(x, y)$ on $T(M^n(c) \setminus \{0\})$ which are positive homogeneous of degree r in the y^i's and whose h-gradient has compact support are given by $f = aE^{r/2}$, $a \in \mathbb{R}$. The image of a totally geodesic immersion of a Finsler space in a Landsberg space M^{n+p} is not contained in any h-convex supporting set of M^{n+p}.

2.1 Conformal properties

Two Finsler manifolds (M, F) and $(M, \widetilde{F})$ are conformally equivalent if there exists a positive smooth function $\varphi : TM \to \mathbb{R}$, called the scale function, such that $\widetilde{g} = \varphi g$. Aikou [2] used the Weyl structure of a conformal class determined by the associated Riemannian metric to characterise when a Finsler manifold is conformal to a Berwald manifold; Vincze [101] showed that in this case the exterior derivative of the scale function is closed and exact, cf also Tamássy [94]. Aikou [3] treats the complex case.

By Hashiguchi's theorem [32], cf also [33], a Landsberg space remains a Landsberg space under any conformal change of metric if and only if its T-tensor field vanishes identically. Matsumoto [63] gave a necessary and sufficient condition for the Berwald spaces property to persist under conformal change of metrics; in the case of a 2-dimensional Berwald space this condition is if and only if it has constant main scalar. Ikeda [40] provided criteria for conformal flatness of Finsler spaces, developing further the method and theorem of Kikuchi [46]. Yang [103] studied the conformal changes between two (a, β)-metric Finsler spaces.

Izumi [43] introduced the notion of a curve in a Finsler space F^n as a geodesic circle if its first and second curvatures are constant and zero, respectively. This definition depends on the choice of the connection so various cases were considered. A conformal transformation $(*)$ $F \to F' = \exp[\sigma(x)]F$ is concircular if it preserves geodesic circles. Necessary and sufficient conditions for concircularity were derived for the various connections in terms of systems of partial differential equations to be satisfied by the conformal function $\sigma(x)$. Subsequently, Izumi [44] called the conformal transformation $(*)$ h-conformal when, for the Finsler objects

$$2g_{hj} = \partial^2 F/\partial y^h \partial y^j, \quad \text{and} \quad 2C_{hjk} = \partial g_{hj}/\partial y^k,$$

his h-condition is satisfied, namely:

$$(n-1)C^h_{ij}\sigma_h = C^h \sigma_h h_{ij}, \quad \text{where} \quad \sigma_h = \partial\sigma/\partial x^h, \quad C_h = C^j_{hj}, \quad h_{ij} = g_{ij} - y_i y_j/F^2.$$

He gave a geometric interpretation of this condition and some h-conformal invariants are displayed, with necessary and sufficient conditions under which F^n is h-conformally flat, that is, h-conformal to a Minkowskian space. Singh and Gupta [91] studied conformal and h-conformal transformation in special Finsler spaces, including the Landsberg case.

Prasad and Dwivedi [79] studied conformal change in three-dimensional Finsler spaces, providing a three-dimensional Finsler space which is conformal to a Berwald space or Landsberg space. Prasad and Dwivedi [78] studied conformal changes: $L(x,y) \to \overline{L}(x,y) = e^{\alpha(x)}L(x,y)$ and the associated changes of Cartan connection and Berwald connection; they investigated the conditions under which m-th root metric Berwald spaces, $\mathbb{S}^3$-like spaces and Landsberg spaces are preserved by a conformal change of the metric.

2.2 Necessary and sufficient conditions for (M, F) to be Landsberg

We collect some characterising conditions from the literature.

1. Along every curve c the parallel translation $P_c\colon (T_{c(a)}M, g_{c(a)}) \to (T_{c(b)}M, g_{c(b)})$ is an isometry between the Riemannian spaces [37].

2. The vertical foliation in (TM, G) is totally geodesic, with [1]

$$G = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}[dx^i \otimes dx^j + \theta^i \otimes \theta^j]$$

$$\theta^i := dy^i + N^i_m dx^m.$$

3. Each fibre is a totally geodesic submanifold in the total space TM with a Sasaki-type metric [1].

4. The infinitesimal h-mapping

$$(x^i_0, l^i_0) \mapsto (x^i = x^i_0 + dx^i_0, l^i = l^i_0 - l^j_0 {\Gamma^*}_j{}^i{}_k(x_0, l_0)dx^k_0)$$

   is affine or an isometry [104]. Shen proved that a Finsler space is Berwald if and only if along every curve parallel translation is an isometry between the associated Minkowski spaces.

5. In a Finsler space with a generalized Randers metric

$$L^2(\alpha, \beta) = c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2$$

(where $\alpha^2 = a_{ij}(x)y^i y^j$ may be any quadratic form, $\beta = b_i(x)y^i$ and c_1, c_2, c_3 are nonzero constants) b_i is a Killing vector field with constant length.

6. For dimension 2, every Landsberg space is a Berwald space.

2.3 Landsberg Problems

We mention some areas that seem still to offer challenges in the study of Landsberg spaces.

Local projectivity Characterization of locally projectively Berwald spaces; see Bácsó [12] for a summary. Shen [87] points out that for locally projectively flat Berwald metrics, the non-trivial case is in dimension two, because locally projectively flat metrics are of scalar flag curvature and Berwald metrics are Landsberg metrics. Bacso-Matsumotos result [12] is a consequence of Numata [70] in dimension greater than two.

Berwald equivalence Every Berwald space is a Landsberg space; however, it has been a long-standing problem to discover if the converse is true; see eg Shen [85] and the new work of Muzsnay [69]. Asanov [10] provides an example of a (singular) Landsberg space that is not Berwald. Shen [88] has since provided a new study of Landsberg (α, β) metrics and shows that a regular such metric is Landsberg if and only if it is a Berwald metric; relaxing regularity allowed him to construct a family of Landsberg metrics that are not Berwald.

Conformal properties Characterization of conformal classes, eg conformal flatness. Prasad and Dwivedi [78] investigated the conditions under which special Finsler spaces with m-th root metric (i.e., Berwald spaces, $\mathbb{S}^3$-like spaces and Landsberg spaces) are preserved by a conformal change of the metric.

Jacobi fields Characterization of Jacobi vector fields. Hassan [35, 36] studied sprays and Jacobi fields in Finsler spaces; Tamin [95] discussed some Randers manifolds and Crampin [27] provided a covariant form of the Lagrangian second variation formula and showed that each of the four standard connections encountered in Finsler geometry produces the same result.

Harmonic maps See Dragomir [30] for some work in this area.

Local flow diffeomorphisms isometric, harmonic.

Warped products Udriste [98] and Kozma et al. [49] have begun the extension of the Riemannian methods to Finsler spaces. See also Asanov [8, 9] for study of the Finsler case $M \times \mathbb{R}$.

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